MINIMAL SYSTEMS OF BINOMIAL GENERATORS AND THE INDISPENSABLE COMPLEX OF A TORIC IDEAL

HARA CHARALAMBOUS, ANARGYROS KATSABEKIS, AND APOSTOLOS THOMA

Abstract. Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{Z}^n$ be a vector configuration and $I_A \subset K[x_1, \ldots, x_m]$ its corresponding toric ideal. The paper consists of two parts. In the first part we completely determine the number of different minimal systems of binomial generators of I_A. We also prove that generic toric ideals are generated by indispensable binomials. In the second part we associate to A a simplicial complex $\Delta_{\mathrm{ind}(A)}$. We show that the vertices of $\Delta_{\mathrm{ind}(A)}$ correspond to the indispensable monomials of the toric ideal I_A, while one dimensional facets of $\Delta_{\mathrm{ind}(A)}$ with minimal binomial A-degree correspond to the indispensable binomials of I_A.

1. Introduction

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be a vector configuration in $\mathbb{Z}^n$ such that the affine semigroup $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ is pointed. Recall that $\mathbb{N}A$ is *pointed* if zero is the only invertible element. Let K be a field of any characteristic; we grade the polynomial ring $K[x_1, \ldots, x_m]$ by setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the *A-degree* of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$\deg_A(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A.$$

The *toric ideal* I_A associated to A is the prime ideal generated by all the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}})$ (see [12]). For such binomials, we define $\deg_A(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}) := \deg_A(\mathbf{x}^{\mathbf{u}})$.

In general it is possible for a toric ideal I_A to have more than one minimal system of generators. We define $\nu(I_A)$ to be the number of different minimal systems of binomial generators of the toric ideal I_A, where the sign of a binomial does not count. A recent problem arising from Algebraic Statistics, see [13], is when a toric ideal possesses a unique minimal system of binomial generators, i.e. $\nu(I_A) = 1$. To study this problem Ohsugi and Hibi introduced in [9] the notion of indispensable binomials while Aoki, Takemura and Yoshida introduced in [1] the notion of indispensable monomials. Moreover in [9] a necessary and sufficient condition is given for toric ideals associated with certain finite graphs to possess unique minimal systems of binomial generators. We recall that a binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$ is *indispensable* if every system of binomial generators of I_A contains B or $-B$, while a monomial

1991 *Mathematics Subject Classification.* 13F20, 05C99.

Key words and phrases. toric ideal, minimal systems of generators, indispensable monomials, indispensable binomials.

This research was co-funded by the European Union in the framework of the program “Pythagoras” of the “Operational Program for Education and Initial Vocational Training” of the 3rd Community Support Framework of the Hellenic Ministry of Education.

$\mathbf{x}^{\mathbf{u}}$ is *indispensable* if every system of binomial generators of I_A contains a binomial B such that the $\mathbf{x}^{\mathbf{u}}$ is a monomial of B.

In this article we use and extend ideas-techniques developed by Diaconis, Sturmfels (see [6]) and Takemura, Aoki (see [13]) to study minimal systems of generators of the toric ideal I_A and also investigate the notion of the indispensable complex of I_A, denoted by $\Delta_{\mathrm{ind}(A)}$. In section 2, we construct graphs $G(\mathbf{b})$, for every $\mathbf{b} \in \mathbb{N}A$, and use them to provide a formula for $\nu(I_A)$. We give criteria for a toric ideal to be generated by indispensable binomials. In section 3 we determine a large class of toric ideals, namely generic toric ideals, which have a unique minimal system of binomial generators. In Section 4 we define $\Delta_{\mathrm{ind}(A)}$ and we show that this complex determines the indispensable monomials and binomials. As an application we characterize principal toric ideals in terms of $\Delta_{\mathrm{ind}(A)}$.

2. The number of minimal generating sets of a toric ideal

Let $A \subset \mathbb{Z}^n$ be a vector configuration so that $\mathbb{N}A$ is pointed and $I_A \subset K[x_1, \ldots, x_m]$ its corresponding toric ideal. A vector $\mathbf{b} \in \mathbb{N}A$ is called an *Betti A-degree* if I_A has a minimal generating set containing an element of A-degree $\mathbf{b}$. The Betti A-degrees are independent of the choice of a minimal generating set of I_A, see [3, 8, 12]. The *A-graded Betti number* $\beta_{0,\mathbf{b}}$ of I_A is the number of times $\mathbf{b}$ appears as the A-degree of a binomial in a given minimal generating set of I_A and is also an invariant of I_A.

The semigroup $\mathbb{N}A$ is pointed, so we can partially order it with the relation

$$\mathbf{c} \geq \mathbf{d} \Longleftrightarrow \text{ there is } \mathbf{e} \in \mathbb{N}A \text{ such that } \mathbf{c} = \mathbf{d} + \mathbf{e}.$$

For $I_A \neq \{0\}$ the minimal elements of the set $\{\deg_A(\mathbf{x}^{\mathbf{u}}) \mid \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A\} \subset \mathbb{N}A$ with respect to $\geq$ are called *minimal binomial A-degrees*. Minimal binomial A-degrees are always Betti A-degrees but the converse is not true, as Example 2.3 demonstrates. For any $\mathbf{b} \in \mathbb{N}A$ set

$$I_{A,\mathbf{b}} := (\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}}) \gneqq \mathbf{b}) \subset I_A.$$

Definition 2.1. *For a vector $\mathbf{b} \in \mathbb{N}A$ we define $G(\mathbf{b})$ to be the graph with vertices the elements of the fiber*

$$\deg_A^{-1}(\mathbf{b}) = \{\mathbf{x}^{\mathbf{u}} \mid \deg_A(\mathbf{x}^{\mathbf{u}}) = \mathbf{b}\}$$

and edges all the sets $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ whenever $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_{A,\mathbf{b}}$.

The fiber $\deg_A^{-1}(\mathbf{b})$ has finitely many elements, since the affine semigroup $\mathbb{N}A$ is pointed. If $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ are vertices of $G(\mathbf{b})$ such that $\gcd(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}) \neq 1$, then $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ is an edge of $G(\mathbf{b})$. The next proposition follows easily from the definition.

Proposition 2.2. *Let $\mathbf{b} \in \mathbb{N}A$. Every connected component of $G(\mathbf{b})$ is a complete subgraph. The graph $G(\mathbf{b})$ is not connected if and only if $\mathbf{b}$ is a Betti A-degree.*

Example 2.3. Let

$$A = \{(2,2,2,0,0), (2,-2,-2,0,0), (2,2,-2,0,0), (2,-2,2,0,0),$$
$$(3,0,0,3,3), (3,0,0,-3,-3), (3,0,0,3,-3)(3,0,0,-3,3)\}.$$

Using CoCoA, [5] we see that $I_A = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_1^3x_2^3 - x_5^2x_6^2)$. The Betti A-degrees are $\mathbf{b}_1 = (4,0,0,0,0)$, $\mathbf{b}_2 = (6,0,0,0,0)$ and $\mathbf{b}_3 = (12,0,0,0,0)$. We note that $\mathbf{b}_3 = 2\mathbf{b}_2$, so $\mathbf{b}_3$ is not a minimal binomial A-degree. The ideals $I_{A,\mathbf{b}_1}$ and $I_{A,\mathbf{b}_2}$ are zero, while $I_{A,\mathbf{b}_3} = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8)$. The graphs $G(\mathbf{b})$ are

connected for all $\mathbf{b} \in \mathbb{N}A$ except for the Betti A-degrees. In fact $G(\mathbf{b}_1)$ and $G(\mathbf{b}_2)$ consist of two connected components, $\{x_1x_2\}$ and $\{x_3x_4\}$ for $G(\mathbf{b}_1)$, $\{x_5x_6\}$ and $\{x_7x_8\}$ for $G(\mathbf{b}_2)$, while the connected components of $G(\mathbf{b}_3)$ are $\{x_1^3x_2^3, x_1^2x_2^2x_3x_4, x_1x_2x_3^2x_4^2, x_3^3x_4^3\}$ and $\{x_5^2x_6^2, x_5x_6x_7x_8, x_7^2x_8^2\}$.

Let $n_{\mathbf{b}}$ denote the number of connected components of $G(\mathbf{b})$, this means that

$$G(\mathbf{b}) = \bigcup_{i=1}^{n_{\mathbf{b}}} G(\mathbf{b})_i$$

and $t_i(\mathbf{b})$ be the number of vertices of the i-component. The next proposition will be helpful in the sequel.

Proposition 2.4. *An A-degree $\mathbf{b}$ is a minimal binomial A-degree if and only if every connected component of $G(\mathbf{b})$ is a singleton.*

Proof. If $\mathbf{b}$ is a minimal binomial A-degree, then $I_{A,\mathbf{b}} = \{0\}$ and every connected component of $G(\mathbf{b})$ is a singleton. Suppose now that $\mathbf{b}$ is not minimal, i.e. $\mathbf{c} \lneqq \mathbf{b}$ for some minimal binomial A-degree $\mathbf{c}$. Thus there is a binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$, with $\deg_A(B) = \mathbf{c}$, and a monomial $\mathbf{x}^{\mathbf{a}} \neq 1$ such that $\mathbf{b} = \mathbf{c} + \deg_A(\mathbf{x}^{\mathbf{a}})$. Therefore $\mathbf{x}^{\mathbf{a}+\mathbf{u}}$, $\mathbf{x}^{\mathbf{a}+\mathbf{v}}$ are vertices of $G(\mathbf{b})$ and belong to the same component of $G(\mathbf{b})$ since $\mathbf{x}^{\mathbf{a}+\mathbf{u}} - \mathbf{x}^{\mathbf{a}+\mathbf{v}} = \mathbf{x}^{\mathbf{a}}B \in I_{A,\mathbf{b}}$. □

Let $\mathcal{G} \subset I_A$ be a set of binomials. We recall the definition of the graph $\Gamma(\mathbf{b})_{\mathcal{G}}$, [6], and a criterion for $\mathcal{G}$ to be a generating set of I_A, Theorem 2.5. Let $\Gamma(\mathbf{b})_{\mathcal{G}}$ be the graph with vertices the elements of $\deg_A^{-1}(\mathbf{b})$ and edges the sets $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ whenever the binomial

$$\frac{(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}})}{\gcd(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})} \quad \text{or} \quad \frac{(\mathbf{x}^{\mathbf{v}} - \mathbf{x}^{\mathbf{u}})}{\gcd(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}})}$$

belongs to $\mathcal{G}$. In [6] the following theorem was proved.

Theorem 2.5. [6] *$\mathcal{G}$ is a generating set for I_A if and only if $\Gamma(\mathbf{b})_{\mathcal{G}}$ is connected for all $\mathbf{b} \in \mathbb{N}A$.*

We consider the complete graph $\mathcal{S}_{\mathbf{b}}$ with vertices the connected components $G(\mathbf{b})_i$ of $G(\mathbf{b})$, and we let $T_{\mathbf{b}}$ be a spanning tree of $\mathcal{S}_{\mathbf{b}}$; for every edge of $T_{\mathbf{b}}$ joining the components $G(\mathbf{b})_i$ and $G(\mathbf{b})_j$, we choose a binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ with $\mathbf{x}^{\mathbf{u}} \in G(\mathbf{b})_i$ and $\mathbf{x}^{\mathbf{v}} \in G(\mathbf{b})_j$. We call $\mathcal{F}_{T_{\mathbf{b}}}$ the collection of these binomials. Note that if $\mathbf{b}$ is not a Betti A-degree, then $\mathcal{F}_{T_{\mathbf{b}}} = \emptyset$.

Theorem 2.6. *The set $\mathcal{F} = \cup_{\mathbf{b} \in \mathbb{N}A} \mathcal{F}_{T_{\mathbf{b}}}$ is a minimal generating set of I_A.*

Proof. First we will prove that $\mathcal{F}$ is a generating set of I_A. From Theorem 2.5 it is enough to prove that $\Gamma(\mathbf{b})_{\mathcal{F}}$ is connected for every $\mathbf{b}$. We will prove the theorem by induction on $\mathbf{b}$. If $\mathbf{b}$ is a minimal binomial A-degree, the vertices of $\Gamma(\mathbf{b})_{\mathcal{F}}$, which are also the vertices and the connected components of $G(\mathbf{b})$, and the tree $T_{\mathbf{b}}$ gives a path between any two vertices of $G(\mathbf{b})$. Next, let $\mathbf{b}$ be non-minimal binomial A-degree. Suppose that $\Gamma(\mathbf{b})_{\mathcal{F}}$ is connected for all $\mathbf{c} \lneqq \mathbf{b}$ and let $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ be two vertices of $\Gamma(\mathbf{b})_{\mathcal{F}}$. We will show that there is a path between these two vertices. We will consider two cases, depending on whether the vertices are in the same connected component of $G(\mathbf{b})$ or not.

(1) If $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ are in the same component $G(\mathbf{b})_i$ of $G(\mathbf{b})$, then $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \sum_i \mathbf{x}^{\mathbf{d}_i}(\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i})$ where $\mathbf{x}^{\mathbf{u}_i}$, $\mathbf{x}^{\mathbf{v}_i}$ have A-degree $\mathbf{b}_i \lneqq \mathbf{b}$. From the inductive

hypothesis $\Gamma(\mathbf{b}_i)_{\mathcal{F}}$ is connected and there is a path from $\mathbf{x}^{\mathbf{u}_i}$ to $\mathbf{x}^{\mathbf{v}_i}$. This gives a path from $\mathbf{x}^{\mathbf{d}_i}\mathbf{x}^{\mathbf{u}_i}$ to $\mathbf{x}^{\mathbf{d}_i}\mathbf{x}^{\mathbf{v}_i}$ and joining these paths we find a path from $\mathbf{x}^{\mathbf{u}}$ to $\mathbf{x}^{\mathbf{v}}$ in $\Gamma(\mathbf{b})_{\mathcal{F}}$.

(2) If $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ belong to different components of $G(\mathbf{b})$ we use the tree $T_{\mathbf{b}}$ to find a path between the two components. In each component we use the previous case and/or the induction hypothesis to move between vertices if needed. The join of these paths provides a path from $\mathbf{x}^{\mathbf{u}}$ to $\mathbf{x}^{\mathbf{v}}$ in $\Gamma(\mathbf{b})_{\mathcal{F}}$.

Next, we will show that no proper subset $\mathcal{F}'$ of $\mathcal{F}$ generates I_A. Let $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{F} \setminus \mathcal{F}'$, and $\deg_A(B) = \mathbf{b}$. Since B is an element of $\mathcal{F}_{T_{\mathbf{b}}}$, it corresponds to an edge $\{G(\mathbf{b})_i, G(\mathbf{b})_j\}$ of $T_{\mathbf{b}}$, and the monomials $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ belong to different components of $G(\mathbf{b})$. Suppose that there was a path $\{\mathbf{x}^{\mathbf{u}_1} = \mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{u}_2}, \dots, \mathbf{x}^{\mathbf{u}_t} = \mathbf{x}^{\mathbf{v}}\}$ in $\Gamma(\mathbf{b})_{\mathcal{F}'}$ joining the vertices $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}}$. Certainly there are monomials $\mathbf{x}^{\mathbf{u}_i}$, $\mathbf{x}^{\mathbf{u}_{i+1}}$ that are in different connected components of $G(\mathbf{b})$. Since $\gcd(\mathbf{x}^{\mathbf{u}_i}, \mathbf{x}^{\mathbf{u}_{i+1}}) \neq 1$ implies that the monomials $\mathbf{x}^{\mathbf{u}_i}$, $\mathbf{x}^{\mathbf{u}_{i+1}}$ are in the same connected component of $G(\mathbf{b})$, we conclude that $\gcd(\mathbf{x}^{\mathbf{u}_i}, \mathbf{x}^{\mathbf{u}_{i+1}}) = 1$ for some i. In this case the binomial $\mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{u}_{i+1}}$ is in $\mathcal{F}'$, has A-degree $\mathbf{b}$, and it corresponds to an edge of $T_{\mathbf{b}}$. By considering these binomials and corresponding edges we obtain a path in $T_{\mathbf{b}}$ joining the components $G(\mathbf{b})_i, G(\mathbf{b})_j$ and not containing $\{G(\mathbf{b})_i, G(\mathbf{b})_j\}$ of $T_{\mathbf{b}}$. This is a contradiction since $T_{\mathbf{b}}$ is a tree. □

The converse is also true; let $\mathcal{G} = \cup_{\mathbf{b}\in\mathbb{N}A}\mathcal{G}_{\mathbf{b}}$ be a minimal generating set for I_A where $\mathcal{G}_{\mathbf{b}}$ consists of the binomials in $\mathcal{G}$ of A-degree $\mathbf{b}$. We will show that $\mathcal{G}_{\mathbf{b}}$ determines a spanning tree $T_{\mathbf{b}}$ of $\mathcal{S}_{\mathbf{b}}$.

Theorem 2.7. *Let $\mathcal{G} = \cup_{\mathbf{b}\in\mathbb{N}A}\mathcal{G}_{\mathbf{b}}$ be a minimal generating set for I_A. The binomials of $\mathcal{G}_{\mathbf{b}}$ determine a spanning tree $T_{\mathbf{b}}$ of $\mathcal{S}_{\mathbf{b}}$.*

Proof. Let $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{G}_{\mathbf{b}}$. The monomials $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ are in different connected components of $G(\mathbf{b})$, otherwise B is not a part of a minimal generating set of I_A. Therefore B indicates an edge in $\mathcal{S}_{\mathbf{b}}$. Let $T_{\mathbf{b}}$ be the union over $B \in \mathcal{G}_{\mathbf{b}}$ of these edges. $T_{\mathbf{b}}$ is tree of $\mathcal{S}_{\mathbf{b}}$, since if $T_{\mathbf{b}}$ contains a cycle we can delete a binomial from $\mathcal{G}$ and still generate the ideal I_A, contradicting the minimality of $\mathcal{G}$. Theorem 2.5 guarantees that the tree $T_{\mathbf{b}}$ is spanning. □

An immediate corollary of Theorems 2.6 and 2.7 concerns the indispensable monomials.

Corollary 2.8. $\mathbf{x}^{\mathbf{u}}$ *is an indispensable monomial of A-degree* $\mathbf{b}$ *if and only if* $\{\mathbf{x}^{\mathbf{u}}\}$ *is a component of* $G(\mathbf{b})$.

We use Theorems 2.6 and 2.7 to compute $\nu(I_A)$, the number of minimal generating sets of I_A. For each $\mathbf{b} \in \mathbb{N}A$ the number of possible spanning trees $T_{\mathbf{b}}$ depends on $n_{\mathbf{b}}$, the number of connected components of $G(\mathbf{b})$. For a given spanning tree $T_{\mathbf{b}}$ the number of possible binomial sets $\mathcal{F}_{T_{\mathbf{b}}}$ (up to a sign) depends on $t_i(\mathbf{b})$, the number of vertices of $G(\mathbf{b})_i$. These numbers determine $\nu(I_A)$. We note that the sum $t_1(\mathbf{b}) + \cdots + t_{n_{\mathbf{b}}}(\mathbf{b})$ is equal to $|\deg_A^{-1}(\mathbf{b})|$, the cardinality of the fiber set $\deg_A^{-1}(\mathbf{b})$. We also point out that $|\mathcal{F}_{T_{\mathbf{b}}}| = n_{\mathbf{b}} - 1$ and that $|\mathcal{F}_{T_{\mathbf{b}}}| = \beta_{0,\mathbf{b}}$, the A-graded Betti number of I_A.

Theorem 2.9. *For a toric ideal I_A we have that*

$$\nu(I_A) = \prod_{\mathbf{b} \in \mathbb{N}A} t_1(\mathbf{b}) \cdots t_{n_\mathbf{b}}(\mathbf{b})(t_1(\mathbf{b}) + \cdots + t_{n_\mathbf{b}}(\mathbf{b}))^{n_\mathbf{b}-2}$$

where $n_\mathbf{b}$ is the number of connected components of $G(\mathbf{b})$ and $t_i(\mathbf{b})$ is the number of vertices of the connected component $G(\mathbf{b})_i$ of the graph $G(\mathbf{b})$.

Proof. Let d_i be the degree of $G(\mathbf{b})_i$ in a spanning tree $T_\mathbf{b}$, i.e. the number of edges of $T_\mathbf{b}$ incident with $G(\mathbf{b})_i$. We have that $\sum_{i=1}^{n_\mathbf{b}} d_i = 2n_\mathbf{b} - 2$. There are

$$\frac{(n_\mathbf{b}-2)!}{(d_1-1)!(d_2-1)!\cdots(d_{n_\mathbf{b}}-1)!}$$

such spanning trees, see for example the proof of Cayley's formula in [7]. For fixed $T_\mathbf{b}$ with degrees d_i, there are $(t_i(\mathbf{b}))^{d_i}$ choices for the monomials for the edges involving the vertex $G(\mathbf{b})_i$. This implies that the number of possible binomial sets $\mathcal{F}_{T_\mathbf{b}}$ is $(t_1(\mathbf{b}))^{d_1} \cdots (t_{n_\mathbf{b}}(\mathbf{b}))^{d_{n_\mathbf{b}}}$. Therefore the total number of all possible $\mathcal{F}_{T_\mathbf{b}}$ is

$$\sum_{d_1+\cdots+d_{n_\mathbf{b}}=2n_\mathbf{b}-2} \frac{(n_\mathbf{b}-2)!}{(d_1-1)!(d_2-1)!\cdots(d_{n_\mathbf{b}}-1)!}(t_1(\mathbf{b}))^{d_1} \cdots (t_{n_\mathbf{b}}(\mathbf{b}))^{d_{n_\mathbf{b}}} =$$

$$= t_1(\mathbf{b}) \cdots t_{n_\mathbf{b}}(\mathbf{b})(t_1(\mathbf{b}) + \cdots + t_{n_\mathbf{b}}(\mathbf{b}))^{n_\mathbf{b}-2}.$$

□

We point out that if $t_i(\mathbf{b}) = 1$ for all i, then the number of possible spanning trees is ${n_\mathbf{b}}^{n_\mathbf{b}-2}$, (Cayley's formula, see [4]). We also note that if $n_\mathbf{b} = 1$, for some $\mathbf{b} \in \mathbb{N}A$, then the factor $t_1(\mathbf{b})(t_1(\mathbf{b}))^{-1}$ in the above product has value 1. Thus the contributions to $\nu(I_A)$ come only from Betti A-degrees $\mathbf{b} \in \mathbb{N}A$. On the other hand we have a unique choice for a generator of degree $\mathbf{b}$ when $n_\mathbf{b} = 2$ and $t_1(\mathbf{b}) = t_2(\mathbf{b}) = 1$. Thus in these cases $G(\mathbf{b})$ consists of two isolated vertices and by Proposition 2.4, $\mathbf{b}$ is minimal. These remarks prove the following:

Corollary 2.10. *Let $B = \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in I_A$ with A-degree $\mathbf{b}$. B is indispensable if and only if the graph $G(\mathbf{b})$ consists of two connected components, $\{\mathbf{x}^\mathbf{u}\}$ and $\{\mathbf{x}^\mathbf{v}\}$. Moreover $\mathbf{b}$ is minimal binomial A-degree.*

Corollary 2.11. *Suppose that the Betti A-degrees $\mathbf{b}_1, \ldots, \mathbf{b}_q$ of I_A are minimal binomial A-degrees. Then*

$$\nu(I_A) = (\beta_{0,\mathbf{b}_1} + 1)^{\beta_{0,\mathbf{b}_1}-1} \cdots (\beta_{0,\mathbf{b}_q} + 1)^{\beta_{0,\mathbf{b}_q}-1}.$$

Proof. By Proposition 2.4, the connected components of $G(\mathbf{b}_i)$ are singletons. It follows that $t_j(\mathbf{b}_i) = 1$ and that $n_{\mathbf{b}_i} = \sum t_j(\mathbf{b}_i)$. Moreover $\beta_{0,\mathbf{b}_i} = |\mathcal{F}_{\mathbf{b}_i}| = n_{\mathbf{b}_i} - 1$. □

The next theorem provides a necessary and sufficient condition for a toric ideal to be generated by its indispensable binomials. It is a generalization of Corollary 2.1 in [13].

Theorem 2.12. *The ideal I_A is generated by its indispensable binomials if and only if the Betti A-degrees $\mathbf{b}_1, \ldots, \mathbf{b}_q$ of I_A are minimal binomial A-degrees and $\beta_{0,\mathbf{b}_i} = 1$.*

Proof. Suppose that I_A is generated by indispensable binomials, then $\nu(I_A) = 1$ and therefore, from Theorem 2.9, $t_j(\mathbf{b}_i) = 1$ and $n_{\mathbf{b}_i} = 2$, for all j, i. Thus $\beta_{0,\mathbf{b}_i} = 1$. Now Proposition 2.4 together with the fact that $t_j(\mathbf{b}_i) = 1$ implies that all $\mathbf{b}_i$ are minimal binomial A-degrees. □

We point out that the above theorem implies that in the case that a toric ideal I_A is generated by indispensable binomials no two minimal generators can have the same A-degree. We compute $\nu(I_A)$ in the following example.

Example 2.13. Let $A = \{a_0 = k, a_1 = 1, \ldots, a_n = 1\} \subset \mathbb{N}$ be a set of $n+1$ natural numbers with $k > 1$ and $I_A \subset K[x_0, x_1, \ldots, x_n]$, the corresponding toric ideal. The ideal I_A is minimally generated by the binomials $x_0 - x_1^k, x_1 - x_2, \ldots, x_{n-1} - x_n$. The Betti A-degrees are $\mathbf{b}_1 = 1$ and $\mathbf{b}_2 = k$, while the A-graded Betti numbers are $\beta_{0,1} = n - 1$ and $\beta_{0,k} = 1$. Also $G(1)$ consists of n vertices, each one being a connected component, and $G(k)$ has two connected components, the singleton $\{x_0\}$ and the complete graph on the $\binom{k+n-1}{n-1}$ vertices $x_1^k, x_1^{k-1}x_2, \ldots, x_n^k$. Thus

$$\nu(I_A) = n^{n-2}\binom{k+n-1}{n-1}.$$

3. Generic toric ideals are generated by indispensable binomials

Generic toric ideals were introduced in [11] by Peeva and Sturmfels. The term generic is justified due to a result of Barany and Scarf (see [2]) in integer programming theory which shows that, in a well defined sense, almost all toric ideals are generic. Given a vector $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, the *support* of α, denoted by $\mathrm{supp}(\alpha)$, is the set $\{i \in \{1, \ldots, m\} \mid \alpha_i \neq 0\}$. For a monomial $\mathbf{x}^{\mathbf{u}}$ we define $\mathrm{supp}(\mathbf{x}^{\mathbf{u}}) := \mathrm{supp}(\mathbf{u})$. A toric ideal $I_A \subset K[x_1, \ldots, x_m]$ is called *generic* if it is minimally generated by binomials with full support, i.e.,

$$I_A = (\mathbf{x}^{\mathbf{u}_1} - \mathbf{x}^{\mathbf{v}_1}, \ldots, \mathbf{x}^{\mathbf{u}_r} - \mathbf{x}^{\mathbf{v}_r})$$

where $\mathrm{supp}(\mathbf{u}_i) \cup \mathrm{supp}(\mathbf{v}_i) = \{1, \ldots, m\}$ for every $i \in \{1, \ldots, r\}$, see [11]. We will prove that the minimal binomial generating set of I_A is a unique.

Theorem 3.1. *If I_A is a generic toric ideal, then $\nu(I_A) = 1$ and I_A is generated by its indispensable binomials.*

Proof. Let $\{B_1, B_2, \ldots, B_s\}$ be a minimal generating set of I_A of full support where $B_i = \mathbf{x}^{\mathbf{u}_i} - \mathbf{x}^{\mathbf{v}_i}$ and $\mathbf{b}_i = \deg_A(B_i)$. We will show that all $\mathbf{b}_i$ are minimal binomial A-degrees. Suppose that one of them, say $\mathbf{b}_1$ is not minimal and that $\mathbf{b}_j + \mathbf{b} = \mathbf{b}_1$ for $\mathbf{b} \in \mathbb{N}A$. It follows that $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{u}_j}$, $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{v}_j}$ are in the same connected component of $G(\mathbf{b}_1)$, where $\mathbf{x}^{\mathbf{a}}$ is a monomial of A-degree $\mathbf{b}$. Since $\mathrm{supp}(\mathbf{u}_j) \cup \mathrm{supp}(\mathbf{v}_j) = \{1, \ldots, m\}$ it follows that at least one of $\gcd(\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{u}_j})$ or $\gcd(\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{v}_j})$ is not 1 and $\mathbf{x}^{\mathbf{u}_1}$ belongs to the same connected component of $G(\mathbf{b}_1)$ as $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{u}_j}$ and $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{v}_j}$. The same holds for $\mathbf{x}^{\mathbf{v}_1}$. This is a contradiction since $\mathbf{x}^{\mathbf{u}_1}$ and $\mathbf{x}^{\mathbf{v}_1}$ belong to different connected components of $G(\mathbf{b}_1)$.

Next we will show that $\beta_{0,\mathbf{b}_i} = 1$. Suppose that one of them, say $\beta_{0,\mathbf{b}_1} = |\mathcal{F}_{T_{\mathbf{b}_1}}|$ is greater than 1. Since $\mathbf{b}_1$ is minimal, the connected components of $G(\mathbf{b}_1)$ are singletons and $n(\mathbf{b}_1) \geq 3$. It follows that in $T_{\mathbf{b}_1}$, two edges share a vertex, and in $\mathcal{F}_{T_{\mathbf{b}_1}}$ the same monomial will necessarily appear in two of our binomial generators.

Since the generators have full support the other two monomials have the same support, thus they have a nontrivial common factor and are in the same component of $G(\mathbf{b}_1)$, a contradiction. $\square$

Example 3.2. Consider the vector configuration $A = \{20, 24, 25, 31\}$. Using CoCoA, [5], we see that $I_A = (x_3^3 - x_1x_2x_4, x_1^4 - x_2x_3x_4, x_4^3 - x_1x_2^2x_3, x_2^4 - x_1^2x_3x_4, x_1^3x_3^2 - x_2^2x_4^2, x_1^2x_2^3 - x_3^2x_4^2, x_1^3x_4^2 - x_2^3x_3^2)$ and therefore it is a generic ideal. We note that the variety $V(I_A)$ is the generic monomial curve in the affine 4-space A^4 of smallest degree, see Example 4.5 in [11]. By Theorem 3.1 $v(I_A) = 1$ and the above generators are indispensable binomials. It follows that the minimal binomial A-degrees are 75, 80, 93, 96, 110, 112 and 122.

4. The indispensable complex of a vector configuration

Consider a vector configuration $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ in $\mathbb{Z}^n$ with $\mathbb{N}A$ pointed, and the toric ideal $I_A \subset K[x_1, \ldots, x_m]$. In [10] it is proved that a binomial B is indispensable if and only if either B or $-B$ belongs to the reduced Gröbner base of I_A for any lexicographic term order on $K[x_1, \ldots, x_m]$. In [1] it is shown that a monomial M is indispensable if the reduced Gröbner base of I_A, with respect to any lexicographic term order on $K[x_1, \ldots, x_m]$, contains a binomial B such that M is a monomial of B. We are going to provide a more efficient way to check if a binomial is indispensable and respectively for a monomial. Namely we will give a criterion that provides the indispensable binomials and monomials with only the information from one specific generating set of I_A.

We let $\mathcal{M}_A$ be the monomial ideal generated by all $\mathbf{x}^{\mathbf{u}}$ for which there exists a nonzero $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$; in other words given a vector $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$ the monomial $\mathbf{x}^{\mathbf{u}}$ belongs to $\mathcal{M}_A$ if and only if there exists $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbb{N}^m$ such that $\mathbf{v} \neq \mathbf{u}$, i.e. $v_i \neq u_i$ for some i, and $\deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}})$. We note that if $\{B_1 = \mathbf{x}^{\mathbf{u}_1} - \mathbf{x}^{\mathbf{v}_1}, \ldots, B_s = \mathbf{x}^{\mathbf{u}_s} - \mathbf{x}^{\mathbf{v}_s}\}$ is a generating set of I_A then $\mathcal{M}_A = (\mathbf{x}^{\mathbf{u}_1}, \ldots, \mathbf{x}^{\mathbf{u}_s}, \mathbf{x}^{\mathbf{v}_1}, \ldots, \mathbf{x}^{\mathbf{v}_s})$. Let $T_A := \{M_1, \ldots, M_k\}$ be the unique minimal monomial generating set of $\mathcal{M}_A$.

Proposition 4.1. *The indispensable monomials of I_A are precisely the elements of T_A.*

Proof. First we will prove that the elements of T_A are indispensable monomials. Let $\{B_1, \ldots, B_s\}$ be a minimal generating set of I_A. Set $M_j := \mathbf{x}^{\mathbf{u}}$ for $j \in \{1, \ldots, k\}$. Since $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ is in I_A for some $\mathbf{v}$, it follows that there is an $i \in \{1, \ldots, s\}$ and a monomial N of B_i such that N divides $\mathbf{x}^{\mathbf{u}}$ and thus $\mathbf{x}^{\mathbf{u}} = N$.

Conversely consider an indispensable monomial $\mathbf{x}^{\mathbf{u}}$ of I_A and assume that is not an element of T_A then $\mathbf{x}^{\mathbf{u}} = M_j\mathbf{x}^{\mathbf{c}}$ for some $j \in \{1, \ldots, k\}$ and $\mathbf{c} \neq \mathbf{0}$. By our previous argument M_j is indispensable. Without loss of generality we may assume that $B_1 = M_j - \mathbf{x}^{\mathbf{z}}$. If $B_j = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$, then

$$B_j' := \mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{z}} - \mathbf{x}^{\mathbf{v}} = B_j - \mathbf{x}^{\mathbf{c}}B_1 \in I_A$$

and therefore $I_A = (B_1, \ldots, B_{j-1}, B_j', B_{j+1}, \ldots, B_s)$. This way we can eliminate $\mathbf{x}^{\mathbf{u}}$ from all the elements of the generating set of I_A, a contradiction to the fact that $\mathbf{x}^{\mathbf{u}}$ is indispensable. $\square$

Definition 4.2. We define the indispensable complex $\Delta_{\mathrm{ind}(A)}$ to be the simplicial complex with vertices the elements of T_A and faces all subsets of T_A consisting of monomials with the same A-degree.

By Proposition 4.1 the indispensable monomials are the vertices of $\Delta_{\mathrm{ind}(A)}$. The connected components consist of the vertices of the same A-degree and are simplices of $\Delta_{\mathrm{ind}(A)}$, actually are facets. Different connected components have different A-degrees. We compute $\Delta_{\mathrm{ind}(A)}$ in the following example.

Example 4.3. In Example 2.13 we have that $\mathcal{M}_A = (x_0, x_1, \dots, x_n)$ and also the facets of $\Delta_{\mathrm{ind}(A)}$ are $\{x_0\}$ and $\{x_1, \dots, x_n\}$.

It follows easily that whenever $\deg_A(\mathbf{x^u})$ is a minimal binomial A-degree, then $\mathbf{x^u} \in T_A$. The converse is not true in general. Indeed in Example 2.13, x_0 belongs to T_A but $\deg_A(x_0)$ is not minimal. Next we give a criterion that determines the indispensable binomials.

Theorem 4.4. *A binomial $B = \mathbf{x^u} - \mathbf{x^v} \in I_A$ is indispensable if and only if $\{\mathbf{x^u}, \mathbf{x^v}\}$ is a 1-dimensional facet of $\Delta_{\mathrm{ind}(A)}$ and $\deg_A(B)$ is a minimal binomial A-degree.*

Proof. Let $\mathbf{b} = \deg_A(B)$. Suppose that $\{\mathbf{x^u}, \mathbf{x^v}\}$ is a 1-dimensional facet of $\Delta_{\mathrm{ind}(A)}$ and $\mathbf{b}$ is minimal binomial A-degree. By Proposition 2.4, minimality of $\mathbf{b}$ implies that the elements of $\deg_A^{-1}(\mathbf{b})$ which are the vertices of $G(\mathbf{b})$, are vertices of $\Delta_{\mathrm{ind}(A)}$ and the connected components of $G(\mathbf{b})$ are singletons. Since $\Delta_{\mathrm{ind}(A)}$ contains only two vertices of A-degree $\mathbf{b}$, $G(\mathbf{b})$ consists of two connected components, $\{\mathbf{x^u}\}$ and $\{\mathbf{x^v}\}$ and B is indispensable by Corollary 2.10. The other direction is done by reversing the implications. $\square$

Theorem 4.4 shows that the toric ideal I_A of Example 2.13 has no indispensable binomials for $n > 2$. Indeed in this case the indispensable complex of I_A contains no 1-simplices that are facets.

We remark that to check the minimality of the A-degree $\mathbf{b}$ of the binomial $B \in I_A$ it is enough to compare $\mathbf{b}$ with the A-degrees of the vertices of $\Delta_{\mathrm{ind}(A)}$. Thus given any generating set of I_A one can compute T_A and construct the simplicial complex $\Delta_{\mathrm{ind}(A)}$. The elements of T_A are the indispensable monomials and the 1-dimensional facets of $\Delta_{\mathrm{ind}(A)}$ of minimal binomial A-degree are the indispensable binomials.

Example 4.5. Let

$$A = \{(2,1,0), (1,2,0), (2,0,1), (1,0,2), (0,2,1), (0,1,2)\}.$$

Using CoCoA, [5], we see that I_A is minimally generated by: $x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2$. Moreover $T_A = \{M_1 = x_1x_6,\ M_2 = x_2x_4,\ M_3 = x_3x_5,\ M_4 = x_3x_6^2,\ M_5 = x_4^2x_5,\ M_6 = x_2x_6^2,\ M_7 = x_4x_5^2,\ M_8 = x_3^2x_6,\ M_9 = x_1x_4^2,\ M_{10} = x_2^2x_6,\ M_{11} = x_1x_5^2,\ M_{12} = x_1^2x_5,\ M_{13} = x_2^2x_3,\ M_{14} = x_1^2x_4,\ M_{15} = x_2x_3^2, M_{16} = x_2x_3x_6,\ M_{17} = x_1x_4x_5\}$. It follows that $\Delta_{\mathrm{ind}(A)}$ is a simplicial complex on 17 vertices and its connected components are the facets

$$\{M_1, M_2, M_3\}, \{M_4, M_5\}, \{M_6, M_7\}, \{M_8, M_9\},$$
$$\{M_{10}, M_{11}\}, \{M_{12}, M_{13}\}, \{M_{14}, M_{15}\}, \{M_{16}, M_{17}\}.$$

The A-degrees of the components are accordingly

$$(2,2,2), (2,2,5), (1,4,4), (4,1,4),$$

$$(2,5,2),(4,4,1),(5,2,2),(3,3,3).$$

All of them are minimal binomial A-degrees and thus I_A has 7 indispensable binomials corresponding to the 1-dimensional facets. We see that all non zero A-graded Betti numbers equal to 1, except from $\beta_{0,(2,2,2)}$ which equals to 2. From Corollary 2.11 we take that $\nu(I_A)=3$.

The next corollary gives a necessary condition for a toric ideal to be generated by the indispensable binomials.

Corollary 4.6. *Let $A=\{\mathbf{a}_1,\ldots,\mathbf{a}_m\}$ be a vector configuration in $\mathbb{Z}^n$. If I_A is generated by the indispensable binomials, then every connected component of $\Delta_{\mathrm{ind}(A)}$ is 1-simplex.*

Proof. Let $\{B_1,\ldots,B_s\}$ be a minimal generating set of I_A consisting of indispensable binomials $B_i=\mathbf{x}^{\mathbf{u}_i}-\mathbf{x}^{\mathbf{v}_i}$. We note that the monomials of the B_i are all indispensable and form T_A. Thus if a face of $\Delta_{\mathrm{ind}(A)}$ contains $\mathbf{x}^{\mathbf{u}_i}$ it also contains $\mathbf{x}^{\mathbf{v}_i}$. By Theorem 4.4 $\{\mathbf{x}^{\mathbf{u}},\mathbf{x}^{\mathbf{v}}\}$ is a facet of $\Delta_{\mathrm{ind}(A)}$. □

The next example shows that the converse of Corollary 4.6 does not hold.

Example 4.7. We return to Example 2.3. The simplicial complex $\Delta_{\mathrm{ind}(A)}$ consists of only two 1-simplices $\{x_1x_2,x_3x_4\}$, $\{x_5x_6,x_7x_8\}$, the indispensable binomials are $x_1x_2-x_3x_4$, $x_5x_6-x_7x_8$ and $I_A\neq(x_1x_2-x_3x_4,x_5x_6-x_7x_8)$.

When $\Delta_{\mathrm{ind}(A)}$ is a 1-simplex the next proposition shows that I_A is principal and therefore generated by an indispensable binomial.

Proposition 4.8. *The simplicial complex $\Delta_{\mathrm{ind}(A)}$ is a 1-simplex if and only if I_A is a principal ideal.*

Proof. One direction of this Proposition is trivial. For the converse assume that $\Delta_{\mathrm{ind}(A)}=\{\mathbf{x}^{\mathbf{u}_1},\mathbf{x}^{\mathbf{v}_1}\}$ and let $B_1:=\mathbf{x}^{\mathbf{u}_1}-\mathbf{x}^{\mathbf{v}_1}$. We will show that $I_A=(B_1)$. Let $B=\mathbf{x}^{\mathbf{u}}-\mathbf{x}^{\mathbf{v}}$ be the binomial of minimal binomial A-degree such that $B\in I_A\setminus(B_1)$. Since $\mathbf{x}^{\mathbf{u}}=\mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{u}_1}$ and $\mathbf{x}^{\mathbf{v}}=\mathbf{x}^{\mathbf{d}}\mathbf{x}^{\mathbf{v}_1}$, where $\mathbf{x}^{\mathbf{c}}\neq\mathbf{x}^{\mathbf{d}}$, and none of them equals to 1, we have that

$$\mathbf{x}^{\mathbf{v}_1}(\mathbf{x}^{\mathbf{c}}-\mathbf{x}^{\mathbf{d}})=\mathbf{x}^{\mathbf{c}}B_1-B.$$

Therefore $0\neq\mathbf{x}^{\mathbf{c}}-\mathbf{x}^{\mathbf{d}}\in I_A$, while $\deg_A(\mathbf{x}^{\mathbf{c}})\lneqq\deg_A(\mathbf{x}^{\mathbf{u}})$ a contradiction. □

References

[1] S. Aoki, A. Takemura and R. Yoshida, *Indispensable monomials of toric ideals and Markov bases*, preprint 2005.

[2] I. Barany and H. Scarf, *Matrices with identical sets of neighbors*, Mathematics of Operation Research **23** (1998) 863-873.

[3] A. Campillo and P. Pison, *L'idéal d'un semi-group de type fini*, Comptes Rendues Acad. Sci. Paris, Série I, **316** (1993) 1303-1306.

[4] A. Cayley, *A theorem on trees*, Quart.J.Math. **23** (1889) 376-378.

[5] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra, Available at `http://cocoa.dima.unige.it`.

[6] P. Diaconis and B. Sturmfels, *Algebraic algorithms for sampling from conditional distributions*, Ann. Statist., **26** (1) (1998) 363-397.

[7] J.H. van Lint and R.M Wilson, A course in Combinatorics, Cambridge University Press 1992.

[8] E. Miller and B. Sturmfels, Combinatorial Commutative Algebra, Graduate Texts in Mathematics **227** Springer Verlag, New York 2005.
[9] H. Ohsugi and T. Hibi, *Indispensable binomials of finite graphs*, J. Algebra Appl. **4** (2005), no 4, 421-434.
[10] H. Ohsugi and T. Hibi, *Toric ideals arising from contingency tables*, Proceedings of the Ramanujan Mathematical Society's Lecture Notes Series, (2006) 87-111.
[11] I. Peeva and B. Sturmfels, *Generic Lattice Ideals*, J. Amer. Math. Soc. **11** (1998) 363-373.
[12] B. Sturmfels, Gröbner Bases and Convex Polytopes. University Lecture Series, No. 8 American Mathematical Society Providence, R.I. 1995.
[13] A. Takemura and S. Aoki, *Some characterizations of minimal Markov basis for sampling from discrete conditional distributions*, Ann. Inst. Statist. Math., **56** (1)(2004) 1-17.

Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54124, GREECE
E-mail address: `hara@math.auth.gr`

Department of Mathematics, University of Ioannina, Ioannina 45110, GREECE
E-mail address: `akatsabekis@in.gr`

Department of Mathematics, University of Ioannina, Ioannina 45110, GREECE
E-mail address: `athoma@cc.uoi.gr`